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# Implementation of Design Sensitivity Analysis for Nonlinear Elastic Structures

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**Implementation of design sensitivity analysis into MARC, a general nonlinear finite element code, is described. Follower forces and constraints treated via Lagrange multipliers are also included in the implementation. The sensitivity analysis is implemented in the most general way; i.e., arbitrary design criteria and design parameters including shape, sizing, and material property variables can be defined through special user-supplied subroutines. Local derivatives of all of the quantities needed for sensitivity analysis are calculated using the central differences allowing a very general implementation without sacrificing the accuracy. Design sensitivities to multiple parameters are implemented, even though an option for multiple loading does not apply to nonlinear finite element analysis. Also, design sensitivities can be calculated at any load level, whereas the analysis calculations march on as if there was no interruption. Derivation of design sensitivities with constraints such as incompressibility or unilateral frictionless contact indicates that the same steps, as in the implementation of regular displacement based nonlinear finite element method, can be followed without any modifications. A numerical example using the modified finite element code is presented to demonstrate the new capability.**

## I. Introduction

DESIGN sensitivity analysis of nonlinear elastic structures has received much attention during recent years. Reference 1, a good overview paper, quotes most of the references on the subject. Quite often in finite element analysis, contact conditions or other constraints such as incompressibility may arise. Use of the Lagrange multipliers is one of the general approaches for treating such constraints. With this approach, the original stiffness matrix is not modified, but it is augmented by terms related to the constraints. Therefore, the sparsity structure of the linear system to be solved is not changed, which is a good feature of this approach. Design sensitivity analysis of linear static and dynamic systems with general constraints treated via Lagrange multipliers is presented in Refs. 2 and 3, whereas nonlinear elastic systems are treated in Ref. 4. When the constraints are implemented in a finite element code using Lagrange multipliers, the design sensitivity calculations follow the same route as for the regular finite element analysis without any constraints, as shown in this paper. That is, the displacement sensitivities are obtained by multiplying the inverse tangent stiffness matrix by partial derivative of the unbalanced force vector. In this process, design sensitivities of the Lagrange multipliers are also obtained without extra calculations. These sensitivities may be useful in calculation of the reaction force sensitivities, or sensitivities of the mean pressure when incompressibility is enforced.

Finite element analysis of nonlinear structures under large displacement and strain conditions frequently deals with follower forces. One example of such forces is the uniformly distributed surface pressure that is always normal to the surface while it deforms. The results on the accuracy and treatment of design sensitivity analysis with follower forces are discussed in Ref. 5. Since the load-stiffness matrix is not calculated in MARC, a general nonlinear finite element code, the effect of follower forces in analysis as well as sensitivity analysis is included by implementing a user element. Some details of that implementation are given in Ref. 5.

Design sensitivity analysis for nonlinear elastic structures can be implemented in a general finite element code in two different ways: inside the code by its modification or outside the code by using restart and postprocessing of analysis results. The basic advantage of the first method is that one can implement the design sensitivity procedures more efficiently and tailor the code according to the current needs. It is also less cumbersome to include the code modified in this manner in an optimization module. The only advantage of the second approach is that one does not have to have the source code. The programming effort is still considerable because most of the finite element processing routines must be developed. Some of these issues are discussed in Refs. 6 and 7.

The purpose of the present paper is to describe implementation of design sensitivity analysis for constrained elastic systems into a general purpose program for nonlinear finite element analysis. The implementation is done inside the code, and the direct differentiation method is chosen over the adjoint method because it is relatively straightforward and easier to implement. In this approach, sensitivities of the displacements and Lagrange multipliers for the constraints are calculated first. Using these, sensitivities of any response-dependent functional can be evaluated. In the computer implementation, a user-supplied subroutine can be used to program sensitivity calculation for any functional using sensitivities of the displacements and the Lagrange multipliers. The adjoint method needs a more cumbersome procedure because the adjoint variable calculations are dependent on the functional whose sensitivities are desired. If more functionals need to be implemented for their sensitivity calculations or if the current functionals need to be modified, the implementation for the adjoint calculations must be updated. This can be quite cumbersome, especially with large general purpose programs. In the current implementation for nonlinear elastic structures,

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any type of element or a combination of them can be used, constraints of incompressibility or unilateral contact can be specified, and shape, sizing, or material parameters can be used as design variables. A numerical example illustrating the developments is presented. The design sensitivities calculated using the developed software are compared with the ones calculated by forward finite differences. Very good agreement is obtained between the two calculations.

## II. Design Sensitivity Analysis with Constraints

Design sensitivity analysis of constrained elastic systems was also presented in Ref. 4 in which both direct differentiation and adjoint methods were described. A special purpose computer code was developed for both analysis and design sensitivity analysis. In such a case, the implementation is fairly straightforward since the developer can tailor both the analysis and the design sensitivity analysis parts for the best results without any consideration for implementation into a general purpose code. In the present paper, we will discuss only the direct differentiation method for design sensitivity analysis and its implementation aspects since, in our opinion, it is more amenable to implementation in a general purpose existing computer code, as explained earlier. Therefore, we will briefly present the direct differentiation method for design sensitivity analysis of nonlinear elastic structures with constraints more explicitly, as it relates to the implementation. It is important to note that for this class of problems, history of sensitivities is not needed to calculate sensitivities at any point of the loading path; i.e., sensitivities at any load level can be calculated by just solving a linear system of equation. This observation simplifies the numerical implementation.

### Nonlinear Analysis

We write a general constraint $h$ on strains $\epsilon_{ij}$ and displacements $u_i$ as

$$h(\epsilon_{ij},u_i) \leq 0 \tag{1}$$

For incompressibility, $h(\epsilon_{ij}, u_i) = I_3 - 1$, where $I_3$ is the third invariant of the metric tensor $I_3 = \det(\delta_{ij} + 2\epsilon_{ij})$, and the inequality in Eq. (1) becomes equality. For contact constraint, $h(\epsilon_{ij},u_i) = (u_i - u_i^0) \cdot n_i - d$, where $n_i$ are the components of the outward normal to the body and $d$ is the distance between the body and the rigid surface. Multiplying the constraint (1) by a virtual Lagrange multiplier $\delta\lambda$ and adding it to a virtual work equation, we obtain

$$\int_V \sigma_{ij}\delta\epsilon_{ij}\, dV - \int_V f_i\delta u_i\, dV - \int_S T_i\delta u_i\, dS + \int_V \delta\lambda h(\epsilon_{ij},u_i)\, dV = 0 \tag{2}$$

where $\sigma_{ij}$ are the components of the second Piola-Kirchoff stress, $f_i$ the components of the body force, and $T_i$ the components of the surface traction. In Eq. (2) the Lagrange multiplier $\delta\lambda$ is 0 if $h(\epsilon_{ij},u_i) < 0$, that is, if the constraint is inactive.

Equation (2) is solved using load incrementation and Newton-Raphson iterations. To demonstrate the relation between finite element and sensitivity analyses we describe these processes in more detail. Body forces $f_i$, surface tractions $T_i$, and/or prescribed displacements are added in increments. Then, Eq. (2) can be written in the incremental form. In terms of the increments of the sought quantities such as displacements, strains, and stresses, the incremental equation, similar to Ref. 8, is given as

$$\int_V \Delta\sigma_{ij}\delta\Delta\epsilon_{ij}\, dV + \int_V \sigma_{ij}\delta\Delta\epsilon_{ij}\, dV - \int_V f_i^k\delta u_i\, dV - \int_S T_i^k\delta u_i\, dS + \int_V \delta\Delta\lambda h(\epsilon_{ij},u_i)\, dV + \int_V \delta\Delta\lambda\left(\frac{\partial h}{\partial\epsilon_{ij}}\Delta\epsilon_{ij} + \frac{\partial h}{\partial u_i}\Delta u_i\right) dV = 0 \tag{3}$$

where

$$f_i^k = f_i^{k-1} + \Delta f_i^k \qquad T_i^k = T_i^{k-1} + \Delta T_i^k$$

and $\Delta f_i^k$ and $\Delta T_i^k$ are increments of the external forces, and superscript $k$ indicates the load increment number.

The nonlinear equation (3) is solved by using Newton-Raphson iterations, i.e., a linearized version of Eq. (3) is solved a number of times and the residual forces are updated until the change in the displacements or the residual forces, etc. (depending on the chosen convergence criterion) satisfy the specified tolerance. Once convergence has been achieved, the increments of the displacements, strains, and stresses are obtained from Eq. (3) and the displacements, strains, and stresses are updated. Then, external forces are incremented, and the process continues until the specified load is reached.

### Design Sensitivity Analysis

Taking variations $\bar{\delta}$ of Eq. (2) due to a design change, a design sensitivity equation is obtained.

$$\int_V \bar{\delta}\sigma_{ij}\delta\epsilon_{ij}\, dV + \int_{\bar{\delta}V} \sigma_{ij}\delta\epsilon_{ij}\, dV - \int_V \bar{\delta}f_i\delta u_i\, dV - \int_{\bar{\delta}V} f_i\delta u_i\, dV - \int_S \bar{\delta}T_i\delta u_i\, dS - \int_{\bar{\delta}S} T_i\delta u_i\, dS + \int_{\bar{\delta}V} \delta\lambda h(\epsilon_{ij}, u_i)\, dV + \int_V \delta\lambda\left(\frac{\partial h}{\partial\epsilon_{ij}}\bar{\delta}\epsilon_{ij} + \frac{\partial h}{\partial u_i}\bar{\delta}u_i\right) dV = 0 \tag{4}$$

The first and the last terms in Eq. (4) contribute to the tangent stiffness matrix after finite element (FE) discretization, and all of the other terms contribute to the right-hand side vector. However, if follower forces are present in the analysis, the third and fifth terms in Eq. (4) may also contribute to the tangent stiffness matrix because the body forces and surface tractions may depend on displacements. The integrals over volume or surface variations can be computed analytically using the reference volume or domain parametrization concept[9-11] or the concept of material derivative.[12] However, since the integrals are normally calculated by means of numerical integration, the integral variations due to shape change must be calculated through variations of the integration point locations. Therefore, calculation of the integral variations are quite straightforward if the integrals are calculated numerically. This approach is followed in the present implementation. We will point out, however, that the reference volume or domain parametrization concept is used implicitly through isoparametric finite elements.

From Eqs. (3) and (4) it is not difficult to see that after FE discretization, the discrete equations corresponding to Eq. (4) and to the Newton-Raphson iterations for Eq. (3) will be

$$[K]\begin{Bmatrix} \dfrac{d\boldsymbol{u}}{d\boldsymbol{b}} \\ \dfrac{d\lambda}{d\boldsymbol{b}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{b}} \end{Bmatrix} \tag{5}$$

where $b$ is a design parameter and

$$[K]\begin{Bmatrix}\Delta u\\ \Delta\lambda\end{Bmatrix} = \begin{Bmatrix}G\end{Bmatrix} \tag{6}$$

where

$$[K] = \begin{bmatrix} K1 & H \\ H^T & 0 \end{bmatrix}$$

is a typical tangent stiffness matrix if Lagrange multipliers are used with $K1$ being a regular tangent stiffness matrix in nonlinear finite element analysis without Lagrange multipliers and $H$ being the typical part of the tangent stiffness matrix associated with the Lagrange multipliers arising from the last term in Eqs. (3) or (4). Also,

$$\begin{Bmatrix}G\end{Bmatrix} = \begin{Bmatrix}f-r\\ g\end{Bmatrix}$$

is the typical right-hand-side vector with Lagrange multipliers with $f - r$ as the typical unbalanced force for the Newton-Raphson iterations in nonlinear finite element analysis and with $g$ as the typical part of the right-hand-side vector arising from the discretization of constraints.

Looking at Eqs. (5) and (6), we conclude that design sensitivity analysis procedure does not change in principle compared to that with the regular finite element analysis without constraints. All the analyst has to do is to calculate the partial derivative of the right-hand side and use the same tangent stiffness matrix as for analysis. However, we gain more information since in addition to the displacement design sensitivities $\mathrm{d}u/\mathrm{d}b$, the design sensitivities of Lagrange multipliers $\mathrm{d}\lambda/\mathrm{d}b$ are also calculated. We note that in case of contact problems or specified displacements, Lagrange multipliers are the reaction forces. In this case, we obtain design sensitivities of them without any additional cost, whereas in a regular finite element analysis these design sensitivities would be obtained through the use of the tangent stiffness matrix and the displacement design sensitivities.

## III. Implementation

### Inside vs Outside Implementation

Implementation of the described design sensitivity analysis with a general finite element analysis codes can, in principle, follow two routes: inside the program and outside the program. The first route means that the source code is modified, some subroutines may be changed, some may be added, and then the modified subroutines are compiled and linked together to produce a new executable which may have input deck and output results different from the original code. In the second approach, the executable is the same, both input deck and output results stay the same (they may, however, have different meanings), whereas design sensitivities are obtained by modifying loading conditions at the final equilibrium stage and using the results.

Both approaches have certain merits. The first one is more efficient as far as the programming is concerned; since all data bases are available, subroutines can be modified inside instead of calling them a second time and feeding modified data, and thus performing redundant computations. Either direct differentiation or the adjoint method can be implemented. Since the design sensitivity step is essentially a linear analysis, design sensitivities can be calculated with respect to multiple design parameters in a similar way to the multiple load option in the codes designed for linear structural analysis. This procedure is easy to implement internally in the code, although it does not apply to nonlinear finite element analysis. This way, design sensitivity analysis becomes extremely efficient for nonlinear finite element computations because the computational cost for the calculation of multiple design sensitivities will be only a fraction of the cost of the regular finite element analysis. If the finite difference procedure is used, each new design parameter implies an extra finite element analysis with a slightly changed input. Besides, situations occur when loads or enforced displacements are applied in a certain sequence, and design sensitivities are needed at the strain or stress level when only some loads or enforced displacements are applied. Then, the finite element analysis has to march on, and design sensitivities are needed at a new strain or stress level, and so on. One of the examples of such a situation is design sensitivities of cyclic stresses or strains. Such situations are also easily handled inside the code since a separate storage can be allocated for design sensitivities. The major drawbacks of the first method are that the source code has to be available which may be expensive and also an investment in the human resources capable of understanding and modifying the code has to be made. The actual amount of additional coding is minimal since existing subroutines can be used to perform most of the operations needed for design sensitivity analysis. The programming efforts of our implementation of design sensitivity analysis into MARC can be estimated as 6 man months.

The major advantage of the second method is that it does not require the source code. However, implementation becomes inefficient because all of the steps in the implementation inside the code have to be followed in the implementation outside the code with the only difference that internal subroutines cannot be modified and access to databases is not transparent. In the case of the adjoint method, some programming may be avoided, and, based on postprocessing of data and restart capabilities, design sensitivities of the desired quantity sometimes may be calculated.[6] However, in many cases, postdata supplied by the code may not be sufficient for the design sensitivity calculations. An example may be the design sensitivities of Cauchy stress for which displacement gradients are needed but typically are not found in the postdata. Then, some programming via user subroutines (if available) and tapping into the data base is required. In addition, features just discussed, such as design sensitivities at different stress or strain levels, cannot be implemented outside the code unless some awkward programming or artificial unloding is carried out.

### Implementation into MARC

Since the source code was available, based on the preceding discussion, design sensitivity analysis was implemented in MARC which is a general purpose commercial finite element code specifically designed for nonlinear analysis. The subroutines responsible for parsing the input deck were modified to accommodate options related to design sensitivity analysis. To identify design parameters, a user subroutine was provided. Using the design sensitivity input options and the user subroutine, the user can fully define the design sensitivity problem at hand in addition to the finite element analysis problem. Identification of the design parameters through a user subroutine provides a lot of generality but may sometimes require considerable programming efforts if shape variations are considered. However, if frequently used parameters are identified, they may be hardwired in the code as standard input options requiring, perhaps, only limited amount of data.

In the direct differentiation approach, the program does not have to know about the user optimization criterion or constraint functionals for sensitivity calculations. The right-hand side for design sensitivity analysis is calculated based on the knowledge of the unbalanced forces and the design parameter. Then the user, based on the calculated sensitivities of the displacements and the Lagrange multipliers, can calculate sensitivities of the desired quantity. Contrary to what was just described, in the adjoint method the program has to have information about the user objective function and constraints, since they are used in the calculation of the right-hand side or adjoint loads.[13]

The major code development, as seen from Eq. (5), was in calculations of the internal force partial derivative $\partial G/\partial b$.

The vector of the internal forces $\boldsymbol{G}$ is calculated for an element as

$$\boldsymbol{G} = \int_V \boldsymbol{B}^T \hat{\boldsymbol{S}} \, \mathrm{d}V \tag{7}$$

where $\boldsymbol{B}$ is the matrix relating the vector of incremental strain field $\delta\hat{\epsilon}$ to the vector of incremental nodal displacements $\delta\boldsymbol{U}$: $\delta\hat{\epsilon} = \boldsymbol{B}\delta\boldsymbol{U}$; and $\hat{\boldsymbol{S}}$ is the stress vector.

To calculate $\partial \boldsymbol{G}/\partial b$ one has to differentiate Eq. (7) with respect to the design parameter either analytically or numerically. The analytical approach is more precise but in a general purpose code requires calculations of derivatives for every finite element type with different shape functions. This, in fact, requires a creation of a new finite element library for design variation analysis in addition to the regular finite element library, therefore requiring massive programming effort. The numerical approach, which is also called the semianalytical method, uses finite difference approximation to the partial derivative of $\boldsymbol{G}$. This method is less precise and may exhibit certain numerical instabilities depending on the step size. The intrinsic inaccuracies for the semianalytical approach in the case of linear elastic structures were investigated in Ref. 14. It was our experience that such instabilities occur in the case of nonlinear elastic structures, too. However, the situation drastically improves if central finite differences are employed instead of forward differences. It is more expensive computationally, requiring calculation of an additional right-hand side compared to the forward difference method, but still is a fraction of the computational cost of nonlinear analysis as a whole, especially when there are a large number of equilibrium iterations. In our experience, the design sensitivity analysis step added on an average 6% of extra CPU time to nonlinear analysis for every design parameter. The major advantage of the semianalytical approach is in its simplicity and uniformity of implementation procedure for all the element types. In addition, we note that in such implementation, it really does not matter whether design sensitivities are calculated to shape variations, thickness variations as in case of beams or shells, constitutive property variations, etc. They are all handled in the same manner by perturbing nodes, thicknesses, or material constants and feeding the variations into the right-hand-side calculations. Practically, however, shape variations involve more information and more programming effort in the user subroutine since the direction of a perturbation has to be known and accounted for.

The partial derivative of the internal forces $\partial \boldsymbol{G}/\partial b$ was calculated according to the central difference scheme as

$$\frac{\partial \boldsymbol{G}}{\partial b} = \frac{\left( \int_{V(b+\Delta b)} \boldsymbol{B}^T(b+\Delta b)\hat{\boldsymbol{S}}(b+\Delta b)\,\mathrm{d}V - \int_{V(b-\Delta b)} \boldsymbol{B}^T(b-\Delta b)\hat{\boldsymbol{S}}(b-\Delta b)\,\mathrm{d}V \right)}{2\Delta b} \tag{8}$$

where $\Delta b$ is an increment of the design parameter $b$. It is important to note that in the calculations in Eq. (8) the displacement field is not recalculated at $b - \Delta b$ or $b + \Delta b$, since $\partial \boldsymbol{G}/\partial b$ is a partial derivative with respect to $b$. It is seen from Eq. (8) that when the design parameter $b$ is perturbed by the amount of $\Delta b$, the quantities $\boldsymbol{B}$ and $\hat{\boldsymbol{S}}$ have to be recalculated twice for $b - \Delta b$ and for $b + \Delta b$. In addition, the integration is also performed over the perturbed volumes $V(b - \Delta b)$ and $V(b + \Delta b)$. Since internal force calculations according to Eq. (7) make use of numerical integration, perturbed quantities $\boldsymbol{B}$ and $\hat{\boldsymbol{S}}$ have to be recalculated at every integration point. Calculations of the perturbed matrix $\boldsymbol{B}$ are relatively simple: for the already obtained displacement vector, the location of an integration point is perturbed and, according to $b - \Delta b$ or $b + \Delta b$, new $\boldsymbol{B}$ is calculated. The calculations of the perturbed stresses $\hat{\boldsymbol{S}}$ are more involved. Stresses in MARC are calculated incrementally, i.e., stress increment is computed at the end of every load step when convergence is achieved and added to the total stress

$$\hat{S}_i = \hat{S}_{i-1} + \Delta\hat{S}_i$$

where $\hat{S}_i$ is the total stress at the end of the $i$th load step, and $\Delta\hat{S}_i$ is the stress increment for the $i$th load increment. Therefore, to calculate perturbed total stresses $\hat{S}(b - \Delta b)$ and $\hat{S}(b + \Delta b)$, one has to save the perturbed stresses for both $b - \Delta b$ and $b + \Delta b$ at the previous load step. Stresses calculated at the element integration points and as well as the numerical integration in Eq. (7) will be affected by the perturbation of the integration point locations. Based on the authors' experience and some numerical experimentation, the choice of $\Delta b = 0.01b$ provides very good accuracy for both shape and nonshape sensitivity calculations. In the case of shape variations, however, $b$ is the dimension of the smallest finite element that is perturbed.

In the implementation of the follower forces, we followed the approach outlined in Ref. 5. The load-stiffness matrix arising in the formulation was implemented through standard four-noded isoparametric membranes applied to the faces of the solid elements with pressure. As shown in Ref. 5, the inclusion of the load stiffness matrix may provide a significant accuracy improvement for certain problems. In dealing with follower forces, when shape variations are involved, external forces, such as pressure normal to the surface, must be perturbed as well, if the shape variations affect the surface on which the force is applied. Again, as in the case of the internal forces, analytical expressions for these gradients can be obtained which look similar to the load-stiffness matrix. However, it was felt that a more universal approach, with a general purpose finite element code, would be to do it by central differences since it allows unrestrained generality with respect to design parameters, considerably simplifies programming, and yet provides very good accuracy.

Design sensitivities to multiple design parameters can be calculated within one analysis when the desired load level is achieved. This is implemented by performing backsubstitution on multiple right-hand sides because the tangent stiffness matrix is available in its decomposed form. Also, as already discussed, design sensitivity analysis may be performed at various load levels while the nonlinear finite element analysis continues without interruption. This is accomplished by restoring all of the quantities like stresses, strains, and displacements to their current values right after the design sensitivity analysis is done.

As was shown in Sec. II, implementation of such features of nonlinear analysis as hybrid incompressible elements, contact conditions, or any constraints implemented in the code through Lagrange multipliers does not require any additional specific coding.

## IV. Numerical Example

An example is presented to illustrate the design sensitivity calculations with the modified code. The model, shown in Fig. 1, consisted of two hybrid solid eight-noded isoparametric elements with three displacement degrees of freedom at each node and with hydrostatic pressure variable constant over the element domain; and two eight-noded shell elements with three displacement degrees of freedom at the corner nodes and with one rotational degree of freedom at the midside nodes with the vector of rotation parallel to the element

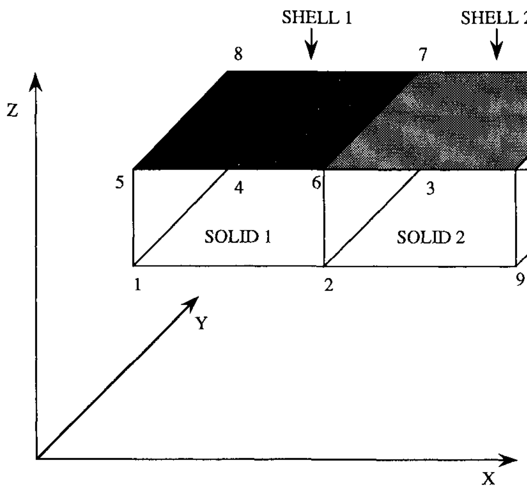


Fig. 1 Finite element model.

**Table 1 Variation comparison with contact conditions—shell thickness variation; Mooney-Rivlin material**

| | Pressure or reaction | Calculated sensitivities | Forward difference sensititivites | Difference, % |
|---|---|---|---|---|
| Solid 1 | −0.7711 | −0.9868 | −0.9803 | 0.6 |
| Solid 2 | −1.4773 | 1.7019 | −1.6897 | 0.7 |
| Contact | 14.9878 | 10.2615 | 10.1933 | 0.7 |

**Table 2 Variation comparison with contact conditions—shape variation; Mooney-Rivlin material**

| | Pressure or reaction | Calculated sensitivities | Forward difference sensititivites | Difference, % |
|---|---|---|---|---|
| Solid 1 | −0.7711 | −0.5506 | −0.5505 | 0.02 |
| Solid 2 | −1.4773 | −1.0295 | −1.0257 | 0.4 |
| Contact | 14.9878 | 11.6288 | 11.5915 | 0.3 |

side. The hydrostatic pressure variable is the Lagrange multiplier to enforce incompressibility.

The Mooney-Rivlin incompressible constitutive law for the solid elements and linear isotropic Hooke's law between second Piola-Kirchoff stress and Green-Lagrange strain were considered. Only one term was retained in the Mooney-Rivlin expansion for strain energy density reducing it to what is sometimes called the Neo-Hookean law. The expression for strain energy density $W$ is

$$W = C(I_1 - 3) + p(I_3 - 1)$$

where $C$ is an experimental constant; $I_1 = 3 + 2\epsilon_{kk}$; $I_3 = \det(\delta_{ij} + 2\epsilon_{ij})$; and $p$ is the hydrostatic pressure. The constant $C$ was 450 psi, the elastic modulus for the shells was 20,000 psi, and the shell thickness was 0.12 in. The shell elements were placed on top of the solids. The dimensions of the structure were 3.1 in. in the $X$ direction, and 2.0 in. in the $Y$ direction. The dimension of the first solid and shell in the $X$ direction was 1.5 in. The dimension of the second solid and shell in the $X$ direction was 1.6 in.

The mesh was fixed at nodes 1, 4, 5, 8, and 16. Then a rigid frictionless plane parallel to the $Z$ axis and equally inclined with respect to the $X$ and $Y$ axes was moved in the negative $X$ direction until it made contact with node 9. Then it was moved an extra 0.1 in. in the negative $X$ direction.

**Table 3 Variation comparison with contact conditions—shell thickness variation; Mooney-Rivlin material**

| | Pressure or reaction | Calculated sensitivities | Forward difference sensititivites | Difference, % |
|---|---|---|---|---|
| Solid 1 | −0.7711 | −0.9868 | −0.9862 | 0.05 |
| Solid 2 | −1.4773 | 1.7019 | −1.7018 | 0.005 |
| Contact | 14.9878 | 10.2615 | 10.2596 | 0.02 |

**Table 4 Variation comparison with contact conditions—shape variation; Mooney-Rivlin material**

| | Pressure or reaction | Calculated sensitivities | Forward difference sensititivites | Difference, % |
|---|---|---|---|---|
| Solid 1 | −0.7711 | −0.5506 | −0.5506 | 0.0 |
| Solid 2 | −1.4773 | −1.0295 | −1.0293 | 0.02 |
| Contact | 14.9878 | 11.6288 | 11.6279 | 0.008 |

Design sensitivities of the mean hydrostatic pressure for both solid elements and the normal contact reaction force between node 9 and the plane were calculated with respect to the shell thickness and to shape variation in which two node locations were perturbed. Nodes 2 and 9 were moved in the negative $Y$ direction proportionally to their distance from node 1, thus perturbing the rectangular shape of the lower solid faces into trapezoids. All of the design sensitivities were calculated within one analysis as was described in the implementation section. The results for both design parameters were compared with the overall forward difference results. The comparisons are given in Tables 1 and 2, for thickness and shape variations, respectively. In Tables 1 and 2, the first two rows provide results for mean pressure and the last row is for the contact force. As is clear from the tables, very good accuracy is obtained compared to the overall finite difference calculation.

To verify further the accuracy of the calculated sensitivities, they were compared with the overall central difference calculations as well, and the results are given in Tables 3 and 4. It is seen that for both shape and nonshape sensitivity calculations, the difference between the results becomes virtually nonexistent indicating that the modified MARC code provides accurate sensitivity results and that the differences in the first example (Tables 1 and 2) are due to numerical error in the forward overall finite difference calculations.

## V. Discussion and Conclusions

Implementation and procedures of design sensitivity analysis of nonlinear elastic solids with constraints and follower forces were described. Description of the procedures implemented into a general purpose nonlinear finite element code was given. In the discussion about the two ways of implementation of design sensitivity analysis into a general finite element code, it was concluded that if the source code is available it is worthwhile to build the design sensitivity analysis inside the code. It is clear from the current implementation that one of the most general and easy ways to implement design sensitivity analysis into a general finite element code without sacrificing accuracy is to use a semianalytical method that uses the central difference scheme.

Derivation of design sensitivities with constraints, such as incompressibility or unilateral frictionless contact implemented through Lagrange multipliers, indicated that the same steps as in implementation of a regular displacement-based nonlinear finite element method need to be followed without any modifications. However, if Lagrange multipliers are employed, additional information, i.e., design sensitivities of the Lagrange multipliers, is obtained without any additional computational cost. This information may be used, for example, in calculation of design sensitivities of the reaction force.

A numerical example illustrating use of design sensitivity analysis with incompressible material and contact conditions implemented via Lagrange multipliers is given. In this example, design sensitivities of hydrostatic pressure and normal contact force were calculated to both thickness and shape variations within one analysis. The results compared very well with finite difference solutions.

The following conclusions are drawn from the present study:

1) Implementation of sensitivity analysis for constrained nonlinear problems follows the same route as for unconstrained problems.

2) Implementation of sensitivity analysis inside the code is preferred if the source code is available.

3) The semianalytical method is preferred for sensitivity implementation in a general finite element code.

4) Very good accuracy in sensitivity calculations can be achieved.

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